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STIEFEL-WHITNEY NUMBERS OF QUATERNIONIC AND RELATED MANIFOLDS

BY
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Abstract. There is considered the image of the symplectic cobordism ring Ω_*^{Sp} in the unoriented cobordism ring N_*. A polynomial subalgebra of N_* is exhibited, with all generators in dimensions divisible by 16, such that the image is contained in the polynomial subalgebra. The methods combine the K-theory characteristic numbers as used by Stong with the use of the Landweber-Novikov ring.

Let M be a closed smooth manifold of dimension k. For n large, M can be embedded in S^{4n+k} in essentially only one way. Let ν denote the normal bundle of the embedding. We say that M is quaternionic if ν has the structure of a quaternionic vector bundle. Stong [11] proved that in dimensions $k<24$ all Stiefel-Whitney numbers of a quaternionic manifold M^k vanish. In his thesis, David Segal [8] extended this to dimensions $k<32$. He also showed the existence of a quaternionic M^{32} unorientedly cobordant to $[RP(2)]^{16}$.

To put the situation more generally, there is the unoriented cobordism ring $N_*=\sum N_k$, and there is a quaternionic cobordism group $\Omega_*^{Sp}=\sum \Omega_k^{Sp}$ [10]. There is a natural homomorphism $\Omega_*^{Sp}\to N_*$, and Im $(\Omega_*^{Sp}\to N_*)$ is the subalgebra of N_* consisting of all cobordism classes represented by quaternionic manifolds M. I am not able to compute this image, but I do give an upper bound to the image and now turn to an outline of the method. In their more widely ranging studies of symplectic cobordism, both Porter [14] and Segal have recently obtained very closely related results.

Let A' denote a graded vector space over Z_2 with basis consisting of all s^R, where R ranges over all sequences $R=(r_1, r_2, \dots)$ of nonnegative integers with $\sum r_k<\infty$ and where $\deg s^R=\sum kr_k$. Landweber [5] and Novikov [7] put a Hopf algebra structure on A'. The product can be defined as composition of operators. Namely each s^R acts on any polynomial algebra $Z_2[x_1, \dots, x_m]$ by

$$\begin{aligned} s^0(x_i) &= x_i, \\ s^{\Delta_k}(x_i) &= x_i^{k+1} \quad \text{where} \quad \Delta_k = (0, \dots, 0, 1, 0, \dots), \\ s^R(x_i) &= 0 \quad \text{if } R \neq 0 \text{ and } R \neq \Delta_k, \\ s^R(yz) &= \sum_{R_1+R_2=R} s^{R_1}(y)s^{R_2}(z). \end{aligned}$$

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The mod 2 Steenrod algebra $\mathscr{S}$ can be considered as a subalgebra of A' by identifying Sq^k with $s^{k\Delta_1}$. There is also the subalgebra T of A' generated by s^{Δ_2} and all the Sq^k. As always, $A'/T^+\cdot A'$ has a natural coproduct where T^+ consists of all sums of elements of T of positive degree. Its dual algebra is a polynomial algebra with generators in each dimension 2^k $(k>1)$, each $2(2k-1)$ where $k\neq 2^j$, and each $2k$ with $k\neq 2^j$. More specifically, we show in §4 that A' is a free left T-module with basis elements s^R for all R with $r_{2^j-1}=0$, $r_{2k-1}=0$ mod 2 for $k\neq 2^j$, $r_{2^k}=0$ mod 2.

Now let MO denote the Thom spectrum of the orthogonal group, so that $\pi_k(MO)$ is the unoriented cobordism group N_k. There is the pairing

$$\tilde{H}^*(MO;Z_2)\otimes N_*\to Z_2$$

sending $s^R\otimes[M]$ into the normal characteristic number mod 2 $s^R[M]$. We may identify $\tilde{H}^*(MO;Z_2)$ with A' and obtain $A'_k\otimes N_k\to Z_2$. According to Thom [12],

$$(A'/\mathscr{S}^+\cdot A')_k\otimes N_k\to Z_2$$

is a dual pairing.

Define $P_k\subset N_k$ to be all cobordism classes $[M]$ of N_k such that $(s^{\Delta_2}s^R)[M]=0$, for all R of degree $k-2$. Then $P=\sum P_k$ is a subalgebra of N_* and P_k is the dual group of $(A'/T^+A')_k$. In §5 we show that we can select a basis $[M^2]$, $[M^4]$, $[M^5]$, ... for the polynomial algebra N_* such that P is the subalgebra generated by all $[M^{2k-1}]^2$ for $k\neq 2^j$, $[M^{2^k}]^2$, $[M^{2k}]$ for $k\neq 2^j$.

Similarly define U to be the subalgebra of A' generated by s^{Δ_2}, $s^{2\Delta_2}$ and all Sq^k. In §4 the structure of A'/U^+A' is computed. In §5 we let $Q_k\subset N_k$ be all $[M]$ with

$$(s^{\Delta_2}s^R)[M]=0,\quad \text{degree } R=k-2,$$
$$(s^{2\Delta_2}s^R)[M]=0,\quad \text{degree } R=k-4.$$

We show that $Q=\sum Q_k$ is given by $Q=P^2$.

Consider now the spectrum MSp coming from the symplectic group. Then $\Omega_*^{Sp}=\pi_*(MSp)$. There is the pairing

$$\tilde{H}^*(MSp;Z_2)\otimes\Omega_*^{Sp}\to Z_2.$$

We can regard $\tilde{H}^*(MSp;Z_2)$ as another copy of the Hopf algebra A', with generators $'S^R$. The two key remarks of §3 which make possible the use of A' are the following: if $[M]\in\Omega_{4k}^{Sp}$ then

$$('S^{\Delta_2}\cdot{}'S^R)[M]=0 \bmod 2,\quad R \text{ of degree } k-2;$$

if $[M]\in\Omega_{8k}^{Sp}$ then

$$('S^{2\Delta_2}\cdot{}'S^R)[M]=0 \bmod 2,\quad \text{degree } R=k-4.$$

These assertions are proved by a very standard use of K-theory, in particular the Adams operations. §§2 and 3 are largely devoted to a proof of these remarks.

The homomorphism $\Omega_*^{Sp} \to N_*$ comes from a map of spectra $g: MSp \to MO$. We have

$$g^*(s^{4R}) = 'S^R, \qquad g^*(s^{R'}) = 0 \quad \text{if } R' \neq 4R.$$

Hence if $[M] \in \operatorname{Im}(\Omega_*^{Sp} \to N_*)$ then

$$(s^{4\Delta_2} \cdot s^R)[M] = 0, \qquad (s^{8\Delta_2} \cdot s^R)[M] = 0,$$
$$s^{R'}[M] = 0, \qquad R' \neq 4R.$$

It follows without difficulty that

$$\operatorname{Im}(\Omega_*^{Sp} \to N_*) \subset Q^4 = P^8.$$

That is, there is a basis $[M^2]$, $[M^4]$, $[M^5]$, $\cdots$ for N_* such that $\operatorname{Im}(\Omega_*^{Sp} \to N_*)$ is contained in the polynomial subalgebra generated by

$$[M^2]^{16}, \quad [M^4]^{16}, \quad [M^5]^{16}, \quad [M^6]^8, \quad \ldots .$$

Here M^{2^k} and M^{2l-1} for $l \neq 2^j$ occur to the power 16, M^{2k} for $k \neq 2^j$ to the power 8. This is one of the two main theorems of the paper.

While about it, it is easy to prove a similar assertion for another natural subalgebra E of N_*. Let E_{2k} consist of all cobordism classes N_{2k} which are represented by weakly almost complex manifolds M for which all Chern numbers

$$c_{2r+1} c_{i_1} \cdots c_{i_p}[M]$$

vanish. We prove that $E \subset P^4$ so that E is contained in the subalgebra generated by

$$[M^2]^8, \quad [M^4]^8, \quad [M^5]^8, \quad [M^6]^4, \quad \ldots .$$

This is the other principal theorem of the paper. We conjecture that $E = P^4$; at least as in §1 one can construct a few nonzero elements of E. In fact, in a later paper we hope to prove $E = P^4$ using methods similar to those of Stong [11].

1. **A few examples.** The purpose of this section is to prove the following.

THEOREM. *There is a closed smooth manifold M^{16n}, unorientedly cobordant to $[RP(2n)]^8$, whose stable tangent bundle*

$$\tau(M) - 16n \in \widetilde{KO}(M)$$

is of the form $2y$ where $y \in \widetilde{KO}(M)$.

Proof. Consider the complex Grassmann manifold

$$M^{16n} = CM(4n, 2)$$

consisting of all 2-dimensional vector subspaces V of C^{4n+2}. There is the bundle ξ associating with each $V \in M$ all $x \in V$, the trivial bundle $4n+2$ associating with V

all $x \in C^{4n+2}$, and $\xi^{\perp}$ associating with V all $x \in V^{\perp}$ where $V^{\perp}$ is the orthogonal complement of V in C^{4n+2}. The tangent bundle to M is [4]

$$\begin{aligned} \tau(M) &= \mathrm{Hom}_C(\xi, \xi^{\perp}) \\ &= \mathrm{Hom}_C(\xi, 4n+2) - \mathrm{Hom}_C(\xi, \xi) \\ &= (4n+2)\bar{\xi} - \mathrm{Hom}_C(\xi, \xi). \end{aligned}$$

There is the operator $\alpha\colon \mathrm{Hom}_C(\xi, \xi) \to \mathrm{Hom}_C(\xi, \xi)$ where $\alpha(\varphi)$ is the adjoint φ^* of φ. Clearly α is a conjugate linear involution. Let

$$A_+ = \{\varphi : \alpha(\varphi) = \varphi\}, \qquad A_- = \{\varphi : \alpha(\varphi) = -\varphi\}.$$

Then

$$\mathrm{Hom}_C(\xi, \xi) = A_+ + A_-$$

and multiplication by i is a real isomorphism between A_+ and A_-. Hence $\mathrm{Hom}_C(\xi, \xi)$ is of the form $2x$ in $\widetilde{KO}(M)$, and $\tau(M) - 16n = 2y$ for some $y \in \widetilde{KO}(M)$.

The equation $[CM(4n, 2)] = [RP(2n)]^8$ follows readily from the equations $[CM(n, k)] = [RM(n, k)]^2$, $[RM(2n, 2k)] = [CM(n, k)]^2$. Here $CM(n, k)$ consists of all k-dimensional complex vector subspaces of C^{n-k} and $RM(n, k)$ of all real k-dimensional real subspaces of R^{n-k}. Each of these follows from the theorem [1]: Let $T\colon M \to M$ be a smooth involution on the closed smooth manifold M^{2n}, with fixed point set F of dimension n; if the normal bundle to F has the same Stiefel-Whitney classes as the tangent bundle of F, then $[M] = [F]^2$.

Consider $R^{2n+2k} = C^{n+k}$, and let $M = RM(2n, 2k)$. There is $T\colon M \to M$ defined by $T(V) = I(V)$ where I is multiplication by i. The fixed point set F is clearly $CM(n, k)$. The inclusion $i\colon F \subset M$ induces $i^*\tau(M)$ and

$$i^*\tau(M) = \mathrm{Hom}_R(i^*\xi, i^*\xi^{\perp}).$$

For $V \in F$, we have V is a complex vector space and

$$\mathrm{Hom}_R(V, V^{\perp}) = \mathrm{Hom}_C(V, V^{\perp}) + \mathrm{Hom}'_C(V, V^{\perp})$$

where the second term denotes all conjugate linear homomorphisms. Now $\mathrm{Hom}_C(i^*\xi, i^*\xi^{\perp})$ is the tangent bundle $\tau(F)$, hence the normal bundle η to F in M is

$$\eta = \mathrm{Hom}'_C(i^*\xi, i^*\xi^{\perp}).$$

Hence

$$\xi(F) = \overline{i^*(\xi)} \otimes_C i^*\xi^{\perp}, \qquad \eta = \overline{i^*\xi} \otimes_C \overline{i^*\xi^{\perp}}.$$

These two bundles have the same Chern classes reduced mod 2, hence they have the same Stiefel-Whitney classes. The second equation follows. The first follows readily from [1, pp. 64–65].

It is possible to put the above construction a little more generally, but I do not know how to take advantage of the added generality. Namely suppose M is a closed smooth manifold whose stable tangent bundle is of the form $2y$ for some $y \in \tilde{K}O(M)$. Let ξ be a smooth complex bundle over M and let $CM(2\xi, 2)$ denote the space of all two-dimensional vector subspaces of fibers of 2ξ. Then $M' = CM(2\xi, 2)$ has stable tangent bundle of the form $2z$ for some $z \in \tilde{K}O(M')$.

Note that we can make M^{16n} weakly almost complex by assigning to its stable tangent bundle the complex structure of the complexification of y. In this complex structure we have $c_{2r+1}(M^{16n})=0$ since on the one hand $c_{2r+1}(M^{16n})$ is torsion and on the other hand $H^*(M^{16n})$ has no torsion. Hence

$$c_{2r+1}c_{i_1}\cdots c_{i_p}[M^{16n}] = 0$$

whether the Chern numbers are those of the tangent bundle or the normal bundle. That is, $[RP(2n)]^8 \in E_{16n}$.

2. **The spectra of Thom spaces.** Let ξ be the canonical complex line bundle over $BU(1)$, and let η be the canonical quaternionic line bundle over $BSp(1)$. There is a map

$$g\colon BU(1) \to BSp(1)$$

with $g^*(\eta) = \xi + \bar{\xi}$. Let $v = \eta - 2$ in $K(BSp(1))$. Then $K(BSp(1))$ is the ring of formal power series $Z[[v]]$. We may take $H^*(BSp(1))$ as $Z[u]$ for $u \in H^4(BSp(1))$, and suppose

$$\mathrm{ch}\, v = u + \text{higher order terms}.$$

Also if ψ^2 is the Adams operation,

$$g^*\psi^2(\eta-2) = \psi^2(\xi+\bar{\xi}-2) = \xi^2+\bar{\xi}^2-2 = (\xi+\bar{\xi}-2)^2+4(\xi+\bar{\xi}-2)$$

so that $\psi^2(v) = 4v+v^2$. More generally we can identify $K^*(BSp(1))$ with $K^*(pt)[[v]]$.

Similarly we can consider $v' = \eta - 1$ as an element of $\tilde{K}O^4(\ ) \approx \tilde{K}Sp(\ )$ and

$$KO^*(BSp(1)) = KO^*(pt)[[v']].$$

The natural homomorphism $KO^*(\ ) \to K^*(\ )$ maps v' into v and v'^n into v^n.

One can next check that

$$H^*(BSp(1)\times\cdots\times BSp(1)) \approx Z[u_1,\cdots,u_n],$$

$$KO^*(BSp(1)\times\cdots\times BSp(1)) \approx KO^*(pt)[[v_1',\ldots,v_n']],$$

$$K^*(BSp(1)\times\cdots\times BSp(1)) \approx K^*(pt)[[v_1,\ldots,v_n]],$$

where $\mathrm{ch}\, v_i = u_i + \text{higher order terms}$, $\psi^2(v_i) = 4v_i + v_i^2$, $KO^*(\ ) \to K^*(\ )$ maps v_i' into v_i.

One can use Dold's theorem [2] on the natural map

$$BSp(1)\times\cdots\times BSp(1) \to BSp(n)$$

to check that $H^*(BSp(n))$, $KO^*(BSp(n))$, $K^*(BSp(n))$ can be identified with the symmetric terms of

$$Z[u_1, \ldots, u_n], \quad KO^*(pt)[[v_1', \ldots, v_n']], \quad K^*(pt)[[v_1, \ldots, v_n]]$$

respectively.

Finally we identify $MSp(n)$ with $BSp(n)/BSp(n-1)$ to obtain an exact sequence

$$\cdots \longrightarrow h^*(MSp(n)) \longrightarrow h^*(BSp(n)) \xrightarrow{f^*} h^*(BSp(n-1)) \longrightarrow \cdots$$

for $h=H$, KO or K. In each case f^* is an epimorphism with kernel the ideal generated by $u_1 \cdots u_n$, $v_1' \cdots v_n'$, $v_1 \cdots v_n$, respectively. Hence we identify $\tilde{H}^*(MSp)$ with the ideal of symmetric polynomials of $Z[u_1, \ldots, u_n]$ generated by $u_1 \cdots u_n$, $\tilde{K}(MSp(n))$ with the ideal of symmetric formal series in $Z[[v_1, \ldots, v_n]]$ generated by $v_1 \cdots v_n$, and $\widetilde{KO}^*(MSp(n))$ with the ideal of symmetric elements of $KO^*[[v_1', \ldots, v_n']]$ generated by $v_1' \cdots v_n'$.

There is the spectrum MSp generated by the $MSp(n)$, and the groups $\tilde{H}^*(MSp)$, $\widetilde{KO}^*(MSp)$, $\tilde{K}^*(MSp)$ defined as inverse limits.

For each sequence $R=(r_1, r_2, \ldots, r_k, \ldots)$ of nonnegative integers with $|R| = \sum r_k < \infty$, there is the element $'S^R$ of $\tilde{H}^*(MSp)$ represented in $\tilde{H}^*(MSp(n))$ by

$$'S^R = \sum u_1^2 \cdots u_{r_1}^2 u_{r_1+1}^3 \cdots u_{r_1+r_2}^3 \cdot u_{r_1+r_2+1}^4 \cdots u_{|R|+1} \cdots u_n$$

for n large.

Define $\|R\| = \sum kr_k$, so that $'S^R \in H^{4\|R\|}(MSp)$.

Hence we identify $\tilde{H}^*(MSp)$ with the free abelian group A generated by all $'S^R$ for all R. Similarly $\tilde{K}(MSp)$ contains a free abelian group A generated by all S^R, where S^R is represented in $\tilde{K}(MSp(n))$ by

$$S^R = \sum v_1^2 \cdots v_{r_1}^2 v_{r_1+1}^3 \cdots v_{r_1+r_2}^3 \cdots v_{|R|+1} \cdots v_n.$$

Moreover ch $S^R = {}'S^R +$ higher order terms.

There is the quaternionic cobordism group Ω_k^{Sp} [10], which may either be regarded as all bordism classes of closed smooth k-manifolds with given quaternionic structure on the stable normal bundle, or as all homotopy classes of maps

$$f\colon S^{4n+k} \to MSp(n), \quad n \text{ large}.$$

Given $[M] \in \Omega_{4k}^{Sp}$ represented by $f\colon S^{4n+4k} \to MSp(n)$ and given $'S^R$, we get the integer

$$'S^R[M] = \langle f^* \, 'S^R, \sigma_{4n+4k} \rangle$$

where σ_{4n+4k} is the orientation class of S^{4n+4k}. Note that $'S^R[M]=0$ unless $\|R\| = k$. Similarly for each $S^R \in \tilde{K}(MSp)$, we get the integer

$$S^R[M] = \langle \text{ch } f^* S^R, \sigma_{4n+4k} \rangle,$$

and $S^R[M]=0$ unless $\|R\| \leq k$. If $\|R\| = k$, then $S^R[M] = {}'S^R[M]$.

The above completes the notation concerning the spectrum MSp. Note that an equivalent discussion holds for MU except for the remarks about KO^*. Namely $\tilde{H}^*(MU)$ can be identified with the free abelian group A generated by all $'S^R$, where $'S^R$ is represented in $H^*(MU(n))$ by

$$'S^R = \sum x_1^2 \cdots x_{r_1}^2 x_{r_1+1}^3 \cdots x_{r_1+r_2}^3 \cdots x_{|R|+1} \cdots x_n.$$

Similarly if $[M] \in \Omega_{2k}^U$ is represented by $f\colon S^{2n+2k} \to MU(n)$, there are the integers $'S^R[M]$, $S^R[M]$ defined as before.

Finally there is the spectrum MO. The cohomology group $\tilde{H}^*(MO; Z_2)$ can be identified with the vector space over Z_2 with basis all s^R where s^R is represented in $\tilde{H}^*(MO(n); Z_2)$ by

$$s^R = \sum z_1^2 \cdots z_{r_1}^2 z_{r_1+1}^3 \cdots z_{r_1+r_2}^3 \cdots z_{|R|+1} \cdots z_n.$$

For $[M] \in N_k$, there is the integer mod 2

$$s^R[M] = \langle f^* s^R, \sigma'_{n+k} \rangle.$$

There are natural maps of spectra

$$MSp \xrightarrow{g} MU \xrightarrow{h} MO.$$

We assume the following facts:

(a) $g^*\colon \tilde{K}(MU) \to \tilde{K}(MSp)$ is an epimorphism,

(b) we have $h^*\colon \tilde{H}^*(MO; Z_2) \to \tilde{H}^*(MU; Z_2)$ given by $h^*(s^{2R}) = {}'S^R \bmod 2$, $h^*(s^R) = 0$ if $R \neq 2R'$,

(c) $g^*\colon \tilde{H}^*(MU; Z_2) \to \tilde{H}^*(MSp; Z_2)$ is given by $g^*('S^{2R}) = {}'S^R \bmod 2$, $g^*('S^R) = 0$ if $R \neq 2R'$.

3. **Divisibility relations among characteristic numbers.** Denote by A a free abelian group with basis elements S^R, one for each sequence $R = (r_1, r_2, \ldots)$ of nonnegative integers with $|R| = \sum r_k < \infty$. Following Landweber [5] and Novikov [7], there is a multiplication defined on A. Each S^R operates as a group homomorphism on any polynomial algebra $Z[x_1, \ldots, x_n]$ with given generators by

$$
\begin{aligned}
S^0(x_i) &= x_i,\\
S^{\Delta_k}(x_i) &= x_i^{k+1} \quad \text{where } \Delta_k = (0, \ldots, 0, 1, 0, \ldots),\\
S^R(x_i) &= 0 \quad \text{if } R \neq 0 \text{ and } R \neq \Delta_k,\\
S^R(yz) &= \sum_{R_1+R_2=R} S^{R_1}(y) \cdot S^{R_2}(z),
\end{aligned}
$$

for all $y, z \in Z[x_1, \ldots, x_n]$.

Given R, R' there exists a unique $T \in A$ with $S^R \circ S^{R'} = T$ on all $Z[x_1, \ldots, x_n]$. In this fashion A becomes an associative ring with unit.

For each nonnegative integer k there is also an operator

$$SQ^k: Z[x_1, \ldots, x_n] \to Z[x_1, \ldots, x_n]$$

defined by

$$\begin{aligned} SQ^0(x_i) &= x_i, \qquad SQ^1(x_i) = x_i^2, \\ SQ^k(x_i) &= 0 \quad \text{for } k > 1, \\ SQ^k(yz) &= \sum_{i+j=k} SQ^i(y) \cdot SQ^j(z). \end{aligned}$$

As operators, we have $SQ^k = S^{(k,0,0,\ldots)}$ and thus we identify SQ^k with $S^{(k,0,\ldots)} \in A$. There are the formulas

$$S^R(x_i^m) = \frac{m!}{r_1!r_2!\cdots(m-|R|)!} x_i^{m+\|R\|}, \qquad SQ^k(x_i^m) = \binom{m}{k} x_i^{m+k}.$$

An element $T \in A$ is uniquely determined by its operator value $T(x_1x_2\cdots x_n)$ in $Z[x_1, \ldots, x_n]$ for n large. Thus

$$\begin{aligned} S^{\Delta_2}(x_1\cdots x_n) &= \sum x_1^3x_2\cdots x_n, \\ (SQ^1\cdot SQ^1 - 2SQ^2)(x_1\cdots x_n) &= 2\sum x_1^3x_2\cdots x_n, \end{aligned}$$

and hence

$$S^{\Delta_2} = \tfrac{1}{2}SQ^1\cdot SQ^1 - SQ^2. \tag{3.1}$$

Similarly

$$\begin{aligned} S^{2\Delta_2}(x_1\cdots x_n) &= \sum x_1^3x_2^3x_3\cdots x_n, \\ (SQ^2\cdot SQ^2 - SQ^3\cdot SQ^1 - SQ^1\cdot SQ^3 + 2SQ^4)(x_1\cdots x_n) &= 4\sum x_1^3x_2^3x_3\cdots x_n, \end{aligned}$$

and hence

$$S^{2\Delta_2} = \tfrac{1}{4}(SQ^2\cdot SQ^2 - SQ^3\cdot SQ^1 - SQ^1\cdot SQ^3 + 2SQ^4). \tag{3.2}$$

We now turn to the pairing

$$A \otimes \Omega_*^{Sp} \to Z, \quad S^R \otimes [M] \to S^R[M]$$

of §2 where $S^R \in \tilde{K}(MSp)$. Let $f: S^{4n+4k} \to MSp(n)$ and let $S^R \in \tilde{K}(MSp(n))$ where $r = \|R\| \leqq k$. Then

$$\begin{aligned} S^R &= \sum v_1^2\cdots v_{r_1+1}^3\cdots v_{|R|+1}\cdots v_n, \\ \psi^2S^R &= \sum (4v_1+v_1^2)^2\cdots(4v_{r_1+1}+v_{r_1+1}^2)^3\cdots \\ &= 4^{n+r}S^R + 4^{n+r-1}(SQ^1\cdot S^R) + \cdots + 4^{n-k+2r}(SQ^{k-r}\cdot S^R) + \cdots. \end{aligned}$$

Also

$$(\psi^2 S^R)[M] = \langle \operatorname{ch} f^*\psi^2 S^R, \sigma_{4n+4k}\rangle$$
$$= 4^{n+k}\langle \operatorname{ch} f^* S^R, \sigma_{4n+4k}\rangle = 4^{n+k} S^R[M].$$

Combining the two equations and dividing by 4^{n-k+2r}, we get

(3.3) $$4^{2k-2r}S^R[M] = 4^{k-r}S^R[M]+\cdots+4(SQ^{k-r-1}\cdot S^R)[M]+(SQ^{k-r}\cdot S^R)[M].$$

Recall here that $[M]\in\Omega_{4k}^{Sp}$ and that $\|R\|=r\leqq k$.

(3.4) Corollary. *For each* $[M]\in\Omega_{4k}^{Sp}$ *and each* S^R *of degree* $r<k$, *we have* $(SQ^{k-r}\cdot S^R)[M]=0 \bmod 4$.

(3.5) *If* $[M]\in\Omega_{8k}^{Sp}$ *and if* $S^R\in\tilde{K}(MSp)$ *is of degree* $r<2k$, *then* $(SQ^{2k-r}\cdot S^R)[M]=0 \bmod 8$.

Proof. Consider $MSp(2n)$, and consider first R with $\|R\|$ odd. Since $\|R\|$ is odd, in $\widetilde{KO}^4(MSp(2n))$ there is the element

$$z = \sum v_1'^2\cdots v_{r_1+1}'^3\cdots v'_{|R|+1}\cdots v'_{2n}$$

and $\widetilde{KO}^*(\ )\to\tilde{K}(\ )$ maps z into S^R. Since $\widetilde{KO}^4(\ )\approx\widetilde{KSp}(\ )$, we may as well suppose that S^R is a quaternionic bundle. Then

$$S^R[M^{8k}] = \langle \operatorname{ch} f^*(S^R), \sigma_{8n+8k}\rangle$$

and since we can consider $f^*(S^R)\in\widetilde{KSp}(S^{8n+8k})$, we get $S^R[M]=0 \bmod 2$. In particular if $\|R\|=2k-1$, (3.3) then gives $12S^R[M^{8k}]=(SQ^1\cdot S^R)[M^{8k}]$ and hence

$$(SQ^1\cdot S^R)[M] = 0 \bmod 8, \qquad \|R\| = 2k-1.$$

The general case now follows from (3.3), for $(SQ^{2k-r-1}\cdot S^R)[M]=0 \bmod 2$, by the case already treated.

We can now convert (3.4) and (3.5) into remarks about cohomology characteristic numbers.

(3.6) *For all* $[M^{4k}]\in\Omega_{4k}^{Sp}$ *and all* $'S^R\in H^{4k-8}(MSp)$ *we have*

$$('S^{\Delta_2}\cdot 'S^R)[M^{4k}] = 0 \bmod 2.$$

Here the product is that of $A=\tilde{H}^*(MSp)$.

Proof. We have

$$('S^{\Delta_2}\cdot 'S^R)[M] = (S^{\Delta_2}\cdot S^R)[M]$$
$$= ((\tfrac{1}{2}SQ^1\cdot SQ^1-SQ^2)\cdot S^R)[M] = 0 \bmod 2$$

by (3.4).

(3.7) *For all* $[M^{8k}]\in\Omega_{8k}^{Sp}$ *and all* $'S^R\in\tilde{H}^{8k-16}(MSp)$ *we have*

$$('S^{2\Delta_2}\cdot 'S^R)[M^{8k}] = 0 \bmod 2.$$

Proof. We have

$$\begin{aligned}('S^{2\Delta_2}\cdot 'S^R)[M] &= (S^{2\Delta_2}\cdot S^R)[M] \\ &= (\tfrac{1}{4}(SQ^2\cdot SQ^2 - SQ^3\cdot SQ^1 - SQ^1\cdot SQ^3 + 2SQ^4)\cdot S^R)[M] \\ &= 0 \bmod 2\end{aligned}$$

by (3.5).

We shall next see that (3.6) applies to a considerably more general class of manifolds M.

Consider $\tilde{H}^*(MU(2n))$ as the ideal of $H^*(BU(2n))$ generated by c_{2n}. In particular, consider $\tilde{H}^*(MU(2n))$ as a module over $H^*(BU(2n))$. The natural map $g: MSp(n) \to MU(2n)$ then has

$$g^*: H^*(MU(2n)) \to H^*(MSp(n))$$

an epimorphism with kernel the submodule generated by $c_1, c_3, \ldots, c_{2k+1}, \ldots$. Hence if

$$g^*: \tilde{K}(MU(2n)) \to \tilde{K}(MSp(n))$$

has $g^*(T)=0$, then $\operatorname{ch} T$ has every term in the submodule generated by $c_1, c_3, \ldots, c_{2k+1}, \ldots$.

Consider now the pairing

$$\tilde{K}(MU) \otimes \Omega^U_{4k} \to Z, \qquad T \otimes [M] \to T[M].$$

It follows readily from the above that if $g^*(T)=0$ and if every Chern number of M^{4k} involving any c_{2r+1} vanishes, then $T[M^{4k}]=0$. Recall that

$$g^*: \tilde{K}(MU) \to \tilde{K}(MSp)$$

is also an epimorphism. Given $S^R \in \tilde{K}(MSp)$, select $T \in \tilde{K}(MU)$ with $g^*(T)=S^R$. Given $[M^{4k}] \in \Omega^U_{4k}$ such that every Chern number of M involving any c_{2r+1} vanishes, then define $S^R[M^{4k}]=T[M^{4k}]$. Note that $S^R[M^{4k}]$ is an integer.

We next see that (3.3) holds for this wider class of manifolds. Recall that $\Omega^{Sp}_* \to \Omega^U_*$ maps Ω^{Sp}_* into the subalgebra W of Ω^U_* consisting of all $[M^{4k}]$ such that every Chern number of $[M^{4k}]$ involving any c_{2r+1} vanishes. Moreover $\Omega^{Sp}_* \to W$ is an isomorphism modulo 2-torsion groups. Hence given $[M^{4k}] \in W$ there is 2^s with $2^s[M^{4k}] \in \operatorname{Im} \Omega^{Sp}_*$. Then (3.3) holds for $2^s M^{4k}$ and hence also for M^{4k}. We can now easily obtain the following.

(3.8) *Suppose that $[M^{4k}] \in \Omega^U_{4k}$ has all Chern numbers involving any c_{2r+1} vanishing. Then for every $'S^R$ in $\tilde{H}^*(MSp)$ with $\|R\|=k$, the number $'S^R[M^{4k}]$ is well defined and* $('S^{\Delta_2}\cdot 'S^R)[M^{4k}]=0 \bmod 2$, $\|R\|=k-2$.

4. The mod 2 algebra $A \otimes Z_2$.

We now set up the machinery to properly utilize (3.7) and (3.8). The starting point is the work of Landweber on the Landweber-Novikov algebra.

Let A' be a vector space over Z_2 with basis consisting of all s^R where R ranges over all sequences $R=(r_1, r_2, \ldots)$ of nonnegative integers with $\sum r_k<\infty$. Every element of A' acts on any polynomial algebra $Z_2[x_1, \ldots, x_n]$ exactly as in §3. This determines an associative ring structure on A', and $A'=A \otimes Z_2$. There are also operators

$$Sq^k\colon Z_2[x_1, \ldots, x_n] \to Z_2[x_1, \ldots, x_n]$$

exactly as before, and we consider $Sq^k=s^{(k,0,\cdots)} \in A'$. Moreover A' is a Hopf algebra with coproduct

$$\psi(s^R) = \sum_{R_1+R_2=R} s^{R_1} \otimes s^{R_2}.$$

Denote by $\mathscr{S} \subset A'$ the subalgebra generated by all the Sq^k, $k \geqq 0$. Reasoning of the type of Milnor (see Landweber again) shows that $\mathscr{S}$ has as additive basis all s^R for all $R=(r_1, r_2, \ldots)$ such that $r_k=0$ whenever k is not of the form 2^j-1.

We can now set the problem of this section. Denote by $T \subset A'$ the subalgebra generated by s^{Δ_2} and by all Sq^k; denote by $U \subset A'$ the subalgebra generated by s^{Δ_2}, by $s^{2\Delta_2}$, and by all Sq^k. Then A' is a left T-module and there is $A'/T^+ \cdot A'$. Moreover A' is a left U-module and there is $A'/U^+ \cdot A'$. In this section we determine the structure of $A'/T^+ \cdot A'$ and $A'/U^+ \cdot A'$.

Linearly order the sequences R by

(i) $R<R'$ if $\|R\|<\|R'\|$,

(ii) $R<R'$ if $\|R\|=\|R'\|$ and if $r_1=r_1', \ldots, r_k=r_k', r_{k+1}>r_{k+1}'$.

Note that this is a well-ordering, since for any k there are only a finite number of R with $\|R\|=k$.

For any R, R' we have

$$\begin{aligned} s^R s^{R'}(x_1 \cdots x_m) &= s^R\left(\sum x_1^2 \cdots x_{r_1'}^2 x_{r_1'+1}^3 \cdots\right) \\ &= a(R, R')s^{R+R'} + \sum_{T>R+R'} a_T s^T, \end{aligned}$$

where

$$a(R, R') = \prod \binom{r_k+r_k'}{r_k} \mod 2.$$

Hence if $a(R, R')=1 \bmod 2$ then

$$s^R s^{R'} = s^{R+R'} + \sum_{T>R+R'} a_T s^T.$$

(4.1) *Suppose that x_i and y_j are sequences of elements of A', where*

$$x_i y_j = s^{R_{i,j}} + \sum_{T>R_{i,j}} a_T s^T.$$

Suppose for $i\neq k$ or for $j\neq l$ that $R_{i,j}\neq R_{k,l}$. The elements x_i, y_j are then linearly independent in A'.

The proof is quite clear. For in any set of elements $R_{i,j}$ there is at least one.

We need now a few computations (see Landweber). Let s_k denote s^{Δ_k} and let $s_{k,k}$ denote $s^{2\Delta_k}$.

$$\begin{aligned}(s_{2p}s_{2q+1})(x_1x_2\cdots x_n) &= s_{2p}\left(\sum x_1^{2q+2}x_2\cdots x_n\right)\\ &= \sum x_1^{2q+2}x_2^{2p+1}\cdots x_n,\\ (s_{2q+1}s_{2p})(x_1\cdots x_n) &= s_{2q+1}\left(\sum x_1^{2p+1}x_2\cdots x_n\right)\\ &= \sum x_1^{2p+2q+2}x_2\cdots x_n+\sum x_1^{2p+1}x_2^{2q+2}\cdots x_n,\end{aligned}$$

and

(1) $$s_{2p}s_{2q+1}+s_{2q+1}s_{2p} = s_{2p+2q+1}.$$

Similarly

(2) $$s_{2p}s_{2p} = s_{4p}$$

so that $s_{2^{k+1}}=(s_2)^{2^k}$. Also

(3) $$s_{2p+1}s_{2q+1} = s_{2q+1}s_{2p+1}, \qquad (s_{2p+1})^2 = 0.$$

Moreover for $k>1$,

$$\begin{aligned}(Sq^n\cdot s_k)(x_1\cdots x_m) &= Sq^n\left(\sum x_1^{k+1}x_2\cdots x_n\right)\\ &= \sum_r\left(\binom{k+1}{r}\sum x_1^{k+r+1}x_2^2\cdots x_{n-r+1}^2x_{n-r+2}\cdots\right)\\ &= \sum\binom{k+1}{r}s_{k+r}\cdot Sq^{n-r}(x_1\cdots x_m)\end{aligned}$$

and

(4) $$Sq^n s_k = \sum\binom{k+1}{r}s_{k+r}\cdot Sq^{n-r}, \qquad k>1.$$

Recall that T is generated by s_2 and the Sq^k. Since $s_1\in\mathscr{S}\subset T$ and $s_2\in T$ we obtain inductively that $s_{2k+1}\in T$ from $s_{2k+1}=s_2s_{2k-1}+s_{2k-1}s_2$. Since $s_{2^j-1}\in\mathscr{S}$, the new information here is that $s_{2k-1}\in T$ for $k\neq 2^j$. Also $s_{2^{k+1}}\in T$.

It follows from (4) that

(5) $$Sq^n\cdot s_{2^k} = s_{2^k}\cdot Sq^n+s_{2^k+1}\cdot Sq^{n-1}+s_{2^{k+1}}\cdot Sq^{n-2^k}+s_{2^{k+1}+1}\cdot Sq^{n-2^k-1},$$

(6) $$\begin{aligned}Sq^n\cdot s_{2k+1} &= \sum\binom{2k+2}{r}s_{2k+r+1}\cdot Sq^{n-r}\\ &= \sum n_{2r}s_{2k+2r+1}\cdot Sq^{n-2r}.\end{aligned}$$

(4.2) *T has as additive basis all*

$$X = (s_2^{\varepsilon_2}s_4^{\varepsilon_4}\cdots s_{2k}^{\varepsilon_{2k}}\cdots)(s_5^{\varepsilon_5}s_9^{\varepsilon_9}s_{11}^{\varepsilon_{11}}\cdots)s^{(r_1,0,r_3,0,0,0,r_7,0,\ldots)}$$

where $\varepsilon_k=0$ or 1 and where all but a finite number of the parameters are zero.

Proof. Letting

$$\begin{aligned} x_i &= (s_2^{\varepsilon_2}s_4^{\varepsilon_4}\cdots)(s_5^{\varepsilon_5}s_9^{\varepsilon_9}\cdots), \\ y_j &= s^{(r_1,0,r_3,0,0,0,r_7,\cdots)}, \\ R_{i,j} &= (r_1, \varepsilon_2, r_3, \varepsilon_4, \varepsilon_5, 0, r_7, \varepsilon_8, \ldots), \end{aligned}$$

we see that $x_i y_j = s^{R_{i,j}}$ + higher terms. Linear independence then follows from (4.1). That every element of T can be written as a linear combination of elements in the form X follows from (1), (2), (3), (5), (6).

(4.3) THEOREM. *Consider A' as a T-module under left multiplication by T. Then A' is a free T-module with basis all elements s^R for all R with $r_{2^j-1}=0$, $r_{2^k}=0 \bmod 2$, $r_{2s-1}=0 \bmod 2$ for $s\neq 2^j$.*

Proof. That A' is a free T-module follows from Milnor-Moore [6], but this is also self-contained. Let

$$\begin{aligned} x_i &= (s_2^{\varepsilon_2}s_4^{\varepsilon_4}\cdots)(s_5^{\varepsilon_5}s_9^{\varepsilon_9}\cdots)s^{(r_1,0,r_3,0,\ldots)}, \\ y_j &= s^{(0,2r_2,0,2r_4,2r_5,r_6,\ldots)}, \\ R_{i,j} &= (r_1, 2r_2+\varepsilon_2, r_3, 2r_4+\varepsilon_4, 2r_5+\varepsilon_5, r_6, \ldots). \end{aligned}$$

Then $x_i y_j = s^{R_{i,j}}$ + larger terms. Hence the elements y_j are linearly independent in A' considered as a T-module, using (4.1). Since every R is of the form $R_{i,j}$ it is clear from induction that the elements $x_i y_j$ generate A' as a vector space, and hence the y_j generate A' as a T-module. The theorem follows.

There is a ring homomorphism $\alpha\colon A' \to A'$ defined by

$$\alpha(s^{2R}) = s^R, \qquad \alpha(s^{R'}) = 0 \qquad \text{if } R' \neq 2R.$$

For consider a polynomial algebra

$$M = Z_2[x_1, \ldots, x_k, \ldots]$$

and in it consider the subalgebra

$$N = \{y^2 \mid y \in M\}.$$

Now each s^R acts on M and $s^{2R}(y^2) = (s^R(y))^2$, $s^{R'}(y^2) = 0$ for $R' \neq 2R$. Thus if $R' \neq 2R$ then $s^{R'}$ acts trivially on N, and there is the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\;s^R\;} & M \\ \approx\big\downarrow & & \big\downarrow\approx \\ N & \xrightarrow{\;s^{2R}\;} & N \end{array}$$

where $M \approx N$ is the squaring map. Thus we let $\alpha(s^{R'})$ be the unique s^R with commutativity holding in

$$\begin{array}{ccc} M & \xrightarrow{s^R} & M \\ \approx\downarrow & & \downarrow\approx \\ N & \xrightarrow{s^{R'}} & N \end{array}$$

In particular, $\alpha(s_{2,2}) = s_2$, $\alpha(Sq^{2k}) = Sq^k$, $\alpha(s_2) = 0$, $\alpha(Sq^{2k+1}) = 0$, and $\alpha(U) = T$ where U is the subalgebra generated by s_2, $s_{2,2}$ and the Sq^k. It follows that α maps $A'/U^+ \cdot A'$ onto $A'/T^+ \cdot A'$.

(4.4) THEOREM. *The homomorphism $\alpha: A' \to A'$ induces an isomorphism*

$$A'/U^+ \cdot A' \approx A'/T^+ \cdot A'$$

of vector spaces.

Proof. We need the following multiplication formulas.

(1) $s_{2,2}s_{4k-2} + s_{4k-2}s_{2,2} = s_{4k+2}$,

(2) $s_{2,2}s_{2,2} + s_{2,2}s_4 + s_2 s_{2,2,2} = s_{4,4}$,

(3) $(s_{2^k,2^k})^2 = s_{2^{k+1},2^{k+1}}$, $k \geqq 2$,

(4) $s_{2k-3,2k-3}s_{4,4} + s_4 s_{2k+1,2k-3} + s_{4,4}s_{2k-3,2k-3} + \binom{2k-2}{2} s_{2k+5,2k-3} = s_{2k+1,2k+1}$.

It follows from (1) that $s_{4k+2} \in U$. Hence also $s_{8k+4} = (s_{4k+2})^2 \in U, \ldots$ and all $s_k \in U$. It follows from (2) that $s_{4,4} \in U$ and from (3) that $s_{2^k,2^k} \in U$ for all k. Finally (4) implies that $s_{2k+1,2k+1} \in U$ for all k.

To prove the theorem, we have only to check that the elements s^R for which $r_{2^j-1} = 0$, $r_{2s-1} = 0 \bmod 4$ for $s \neq 2^j$, $r_{2^k} = 0 \bmod 4$, $r_{2t} = 0 \bmod 2$ for $t \neq 2^j$ generate A' as a U-module. For clearly the epimorphism $A'/U^+ \cdot A' \to A'/T^+ \cdot A'$ maps these elements into a basis.

Fix m and consider all R with $\|R\| = m$. The largest such s^R is s^{Δ_m} and $s^{\Delta_m} \in U^+$. Fix R and assume that for every $R' > R$ with $\|R'\| = m$ that $s^{R'}$ is expressible in terms of the proposed basis elements. Also suppose that every $s^{R''}$ is expressed in terms of the proposed base elements if $\|R''\| < m$. Let $R = (r_1, r_2, \ldots)$.

Case 1. If some $r_{2^k-1} = t \neq 0$ then

$$s^{t\Delta_{2^k-1}} s^{(r_1, \ldots, r_{2^k-2}, 0, r_{2^k}, \ldots)} = s^R + \text{larger terms.}$$

Hence in this case s^R can be suitably expressed.

Case 2. $r_{2^k} = t = 4r + \varepsilon$ where $\varepsilon = 1$, 2 or 3. Taking the case $\varepsilon = 3$ we have

$$s_{2^k} \cdot s^{(r_1, \ldots, r_{2^k-1}, 4r+2, r_{2^k+1}, \ldots)} = s^R + \text{larger terms}$$

and s^R is expressed in terms of the proposed base elements. Similarly if $\varepsilon = 1$.

Taking the case $\varepsilon=2$ we have

$$s_{2^k,2^k}s^{(r_1,\dots,r_{2^k-1},4r,\dots)} = s^R+\text{larger terms}$$

and this case follows.

Case 3. $r_{2k+1}=4r+\varepsilon$ where $\varepsilon=1$, 2 or 3. This goes exactly as Case 2.

Case 4. $r_{2k}=2r+1$, $k\neq 2^j$.

$$s_{2k}s^{(r_1,\dots,r_{2k-1},2r,\dots)} = s^R+\text{larger terms.}$$

Since $s_{2k}\in U^+$, this case follows.

If none of the above hold, then s^R is one of the proposed base elements. The theorem follows.

5. **The main theorems.** Recall that in §2 we have identified $\tilde{H}^*(MO; Z_2)$ with the algebra $A'=A\otimes Z_2$ of §4. If N_* is the unoriented cobordism group, there is also the pairing $A'\otimes N_*\to Z_2$ mapping $s^R\otimes[M]$ into $s^R[M]$. Letting $A'=\sum A'_k$ where A'_k is generated by all s^R with $\|R\|=k$, we have

$$A'_k\otimes N_k\to Z_2.$$

Recalling that $\mathscr{S}\subset A'$ is the Steenrod algebra, we know from Thom that $(\mathscr{S}^+\cdot A')_k\otimes N_k\to 0$ and that *the induced map*

$$(A'/\mathscr{S}^+\cdot A')_k\otimes N_k\to Z_2 \tag{5.1}$$

is a dual pairing.

Consider now the ring $T\supset\mathscr{S}$ of §2. *Define P_k as the subgroup of N_k consisting of all $[M]\in N_k$ such that $(s_2\cdot s^R)[M]=0$ for all s^R of degree $k-2$.* Alternatively, P_k is the annihilator of $(T^+\cdot A')_k$ in $A'\otimes N_k\to Z_2$ and thus there is the dual pairing

$$(A'/T^+\cdot A')_k\otimes P_k\to Z_2.$$

(5.2) *$P=\sum P_k$ is a subalgebra of N_*.*

Proof. If $[M]$, $[M']\in P$, then

$$(s_2\cdot s^R)[M\times M'] = \sum_{R_1+R_2=R}(s_2s^{R_1})[M]\cdot s^{R_2}[M'] + \sum_{R_1+R_2=R}s^{R_1}[M]\cdot(s_2s^{R})[M']$$

and $[M\times M']\in P$.

(5.3) Theorem. *There exist generators $[M^2]$, $[M^4]$, $[M^5],\dots$ for N_* as a polynomial algebra over Z_2 such that P is the polynomial subalgebra with generators $[M^2]^2$, $[M^4]^2$, $[M^6]$, $[M^8]^2$, $[M^9]^2$, $[M^{10}],\dots$. Here every $[M^{2k+1}]$ and every $[M^{2^k}]$ occurs to the power two and every $[M^{2k}]$ with $k\neq 2^j$ to the power one.*

Proof. We know that $(A'/T^+\cdot A')_k$ and P_k are dually paired. In (4.3) the dimension of $(A'/T^+\cdot A')_k$ has been obtained and it agrees with that of the above polynomial subalgebra. Hence we have only to show the existence of a polynomial basis for N_* such that $[M^{2k+1}]^2$, $[M^{2k}]^2 \in P$ and $[M^{2k}] \in P$ for $k \neq 2^j$. In the first cases the choice is arbitrary since

$$(s_2 s^R)[M \times M] = \sum s_2 s^{R_1}[M] \cdot s^{R_2}[M] + \sum s^{R_1}[M] \cdot s_2 s^{R_2}[M] = 0.$$

Thus we have only to show the existence for $k \neq 2^j$ of an $[M^{2k}] \in P_{2k}$ with $s_{2k}[M^{2k}] = 1$. From (4.3) we know that $s_{2k} \notin T^+ \cdot A'$ for $k \neq 2^j$. Since

$$(A'/T^+ \cdot A')_k \otimes P_k \to Z_2$$

is a dual pairing, there exists $[M^{2k}] \in P_{2k}$ with $s_{2k}[M^{2k}] = 1$. The theorem follows.

(5.4) THEOREM. *Define Q_k to be the set of all cobordism classes $[M] \in N_k$ such that*
(i) $(s_2 \cdot s^R)[M] = 0$, *all R with $\|R\| = k-2$, and*
(ii) $(s_{2,2} \cdot s^R)[M] = 0$, *all R with $\|R\| = k-4$.*
Let $Q = \sum Q_k$. Then Q is a subalgebra of N_* and $Q = P^2 = \{x^2 \mid x \in P\}$.

Proof. It is readily checked that Q is a subalgebra, and this is left to the reader. We have a dual pairing

$$(A'/U^+ \cdot A')_k \otimes Q_k \to Z_2$$

and an isomorphism

$$\alpha\colon A'/U^+ \cdot A' \approx A'/T^+ \cdot A'.$$

It is also readily checked that if $M \in P_k$ then $[M]^2 \in Q_{2k}$. The diagram

$$\begin{array}{ccc} P_k & \longrightarrow & \mathrm{Hom}\,[(A'/T^+A')_k, Z_2] \\ \big\downarrow f & & \approx \big\downarrow \alpha' \\ Q_{2k} & \longrightarrow & \mathrm{Hom}\,[(A'/U^+A')_{2k}, Z_2] \end{array}$$

is commutative, where $f(x) = x^2$. Hence f is an isomorphism and $Q_{2k} = (P_k)^2$. The theorem follows.

It follows immediately from (5.3) that $P_{2k+1} = 0$. Hence in (5.4) we may replace (ii) by

(ii)′ $(s_{2,2} \cdot s^R)[M] = 0$ when $\dim M = 2k$, $\|R\| = 2k-4$.

Let E_{2k} denote the subgroup of N_{2k} consisting of all cobordism classes represented by a weakly almost complex manifold M^{2k} such that its Chern numbers satisfy

$$c_{2r+1} c_{i_1} \cdots c_{i_p}[M] = 0, \qquad r = 0, 1, 2, \ldots.$$

Note that this condition is equivalent whether the stable tangent bundle or the normal bundle is used to define Chern classes. Let $E = \sum E_{2k}$.

(5.5) THEOREM. *The subalgebras E, P and* $\mathrm{Im}\,(\Omega_*^{Sp} \to N_*)$ *of N_* satisfy $E \subset P^4$,* $\mathrm{Im}\,(\Omega_*^{Sp} \to N_*) \subset P^8$.

Proof. The map $g: MSp \to MO$ has $g^*(s^{4R}) = {'s^R} \bmod 2$, $g^*(s^{R'}) = 0$ for $R' \neq 4R$. Hence for each $[M]$ in Im $(\Omega_*^{Sp} \to N_*)$ we have $s^{R'}[M]=0$ for $R' \neq 4R$. From (3.6) and (3.7) we have

$$(s^{4\Delta_2} \cdot s^R)[M] = 0, \qquad (s^{8\Delta_2} \cdot s^R)[M] = 0,$$

the second holding in case dim $M = 8k$ and $\|R\| = 8k - 16$.

From $s^{R'}[M]=0$ unless $R' = 4R$ it follows that $[M] = [M']^4$ for some $[M']$. For define $A' \to Z_2$ by $s^R \to s^{4R}[M]$. Alternatively there is $\alpha^2: A' \to A'$ and $\alpha^2(x) \to x[M]$, which is well defined. Then $\alpha^2(Sq^{4n}s^{4R}) = Sq^n \cdot s^R$, hence $Sq^n \cdot s^R \to 0$ and the homomorphism $A' \to Z_2$ kills $\mathscr{S}^+ \cdot A'$. Hence there is an $[M']$ with $s^R[M'] = s^{4R}[M]$, and $[M] = [M']^4$. Then

$$(s^{\Delta_2} \cdot s^R)[M'] = 0, \qquad (s^{2\Delta_2} \cdot s^R)[M'] = 0,$$

the latter holding in case dim $M = 8k$. Then by (5.4) and the remark following it, we have $[M'] \in P^2$ and Im $(\Omega_*^{Sp} \to N_*) \subset P^8$. In exactly the same way using (3.8) we see that $E \subset P^4$.

6. **Remarks and questions.** (a) It can be shown that every $[M] \in P_k$ is a Wall manifold, i.e.

$$w_1^2 w_{i_1} \cdots w_{i_p}[M] = 0, \quad \text{all } i_1, \ldots, i_p.$$

Thus P is contained in the Wall subalgebra W of N_*. Clearly $P \neq W$ since $P_5 = 0$ and $W_5 \neq 0$. It would be interesting to have a characterization of P as a subalgebra of W. A generator for W_6 can also serve as generator for P_6. Hence $[M^6]$ can be taken to be the $CP(2)$ bundle over the Klein bottle used by Wall. I have not tried to obtain models for all the generators M^{2k}.

(b) From §3 one gets the impression that one might try to study divisibility relations between Chern numbers in terms of the algebra A. The advantage is that the Adams operations on $A \subset \tilde{K}(MU)$, when suitably stabilized, can be put in terms of the multiplication of A. Can one understand the Stong results [9] on

$$\tilde{K}(MU) \otimes \Omega_*^U \to Z$$

purely arithmetically? Can one use A to make a general arithmetic of divisibility relations?

(c) In (5.5) it has been shown only that $E \subset P^4$, but it seems safe to conjecture $E = P^4$. Since P^4 is a polynomial algebra with generators of dimensions $16, 32, 40, 24, 64, 72, 40, \ldots$, it would be enough to exhibit generators in E in these dimensions. Generators in dimension $16, 32, 64, \ldots$ were constructed in §1. By messy calculations I have shown the existence of an M^{24}. Hence in dimensions $k < 40$ we have $E_k = (P^4)_k$. Are there polynomial generators of P_*^4 whose stable tangent bundles are complexifications of real bundles? Here the first unanswered dimension is 24.

(d) Is Im $(\Omega_*^{Sp} \to N_*) = P^8$?

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